# BLOCK HYBRID METHOD FOR SOLVING HIGHER ORDER ORDINARY DIFFERENTIAL EQUATION USING POWER SERIES ON IMPLICIT ONE-STEP SECOND DERIVATIVE

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Abstract: Our focus in this research is to developed block method for solving higher order ordinary differential equation using power series on implicit one-step. In order to achieve the aim and objective of this research, we used interpolation, collocation and evaluate a power series approximation at some chosen grid and off-grid points to generate an implicit continuous hybrid one-step method. As requirement of any numerical analyst, the properties of one-step block method was done and results showed that it is consistent, convergent, zero stable and with region of absolutely stable. The method was tested with numerical examples solved using the existing methods and our method was found to give better results when compared with the existing method. Obviously, the solution graphs show the convergence of the method with exact solutions.

**Keywords:** Block method, higher order ODE, interpolation, collocation, power series

## Introduction

Most of the problems in science, mathematical physics and engineering are formulated by differential equations. The solution of differential equations is a significant part to develop the various modeling in science and engineering. There are many analytical methods for finding the solution of ordinary differential equations. But a few numbers of differential equations have analytic solutions where a large numbers of differential equations have no analytic solutions.

In recent years, mathematical modeling of processes in biology, physics and medicine, particular in dynamic problems, cooling of a body and simple harmonic motion has led to significant scientific advances both in mathematics and biosciences (Brauer & Chavez, 2012, Elazzouzi *et al*., 2019).

A differential equation can be classified into ordinary differential equation (ODE), partial differential equation (PDE), stochastic differential equation (SDE), impulsive differential equation (IDE), delay differential equation (DDE), etc. (Stuart & Humphries 1996).

In recent times, the integration of Ordinary Differential Equations (ODEs) is investigated using some kind of block methods. We consider the solution of equation in the form;

$$y''=f(x,y,y'),\ \ y(x_0)=y_0,\ \ y'(x_0)=y'_0 \quad (1.1)$$

Literature has shown that many numerical problems can be modeled into problem (1.1). Though the conventional method for modeling (1.1) is by reducing it tosystem of first order ordinary differential equations. Over the years, different numerical methods have been developed in order to model the solution of equation (1.1). Among these methods are block method, linear multistep method, hybrid method and Rung-Kutta method, (Lambert 1973, Gear 1966, 1971& 1978, Suleiman, 1979& 1989). Recently, some scholars have been made an effort to develop hybrid block method for solving (1.1) directly, among others are Kuboye & Omar (2015), Omar & Abdelrahim (2016), Abdelrahim & Omar (2016), Alkasassbeh & Omar (2017), Skwame *et al*. (2019a, 2019b & 2020).

In literature, some scholars such as Omar (1999) and Olabode (2007) proposed block methods for solving higher order ordinary differential equations directly without reduction to a systems of first order ODEs. Abdelraim *et al*. (2019), Moammad & Omar (2017), Omar & Alkasassbeh (2016), Abdelrahim & Omar (2016) have proposed one-step block hybrid method for the direct solution of second order ordinary differential equationand yield at a good results, their work motivate us to propose block method for solving higher order ordinary differential equation using power series on implicit one-step.

## Materials and Method

We consider a power series approximate solution in the form:

$$y(x)=\sum_{j=0}^{p+q-1} a_j\,x^j \quad (2.1)$$

**where** $p$ and $q$ are number of distinct interpolation and collocation, respectively.

Differentiating ((2.1) twice, yield

$$\sum_{j=0}^{p+q-1} j(j-1)a_j\,x^{j-2}$$

(2.2)

Substituting (2.2) into (1.1) yield

$$\sum_{j=0}^{p+q-1} j(j-1)a_j\,x^{j-2}=f(x,y,y') \quad (2.3)$$

Now, interpolating (2.1) at point $x=x_{n+\frac{1}{8}},x_{n+\frac{3}{8}}$ and collocating (2.3) at $x=x_n,x_{n+\frac{1}{8}},x_{n+\frac{3}{8}},x_{n+\frac{5}{8}},x_{n+\frac{7}{8}},x_{n+1}$ lead to a system of equation written below;

$$AX=B \quad (2.4)$$

$$\begin{pmatrix} 1 & x^1_{n+\frac{1}{8}} & x^2_{n+\frac{1}{8}} & x^3_{n+\frac{1}{8}} & x^4_{n+\frac{1}{8}} & x^5_{n+\frac{1}{8}} & x^6_{n+\frac{1}{8}} & x^7_{n+\frac{1}{8}} \\ 1 & x^1_{n+\frac{3}{8}} & x^2_{n+\frac{3}{8}} & x^3_{n+\frac{3}{8}} & x^4_{n+\frac{3}{8}} & x^5_{n+\frac{3}{8}} & x^6_{n+\frac{3}{8}} & x^7_{n+\frac{3}{8}} \\ 0 & 0 & 2 & 6x^1_n & 12x^2_n & 20x^3_n & 30x^4_n & 42x^4_n \\ 0 & 0 & 2 & 6x^1_{n+\frac{1}{8}} & 12x^2_{n+\frac{1}{8}} & 20x^3_{n+\frac{1}{8}} & 30x^4_{n+\frac{1}{8}} & 42x^4_{n+\frac{1}{8}} \\ 0 & 0 & 2 & 6x^1_{n+\frac{3}{8}} & 12x^2_{n+\frac{3}{8}} & 20x^3_{n+\frac{3}{8}} & 30x^4_{n+\frac{3}{8}} & 42x^4_{n+\frac{3}{8}} \\ 0 & 0 & 2 & 6x^1_{n+\frac{5}{8}} & 12x^2_{n+\frac{5}{8}} & 20x^3_{n+\frac{5}{8}} & 30x^4_{n+\frac{5}{8}} & 42x^4_{n+\frac{5}{8}} \\ 0 & 0 & 2 & 6x^1_{n+\frac{7}{8}} & 12x^2_{n+\frac{7}{8}} & 20x^3_{n+\frac{7}{8}} & 30x^4_{n+\frac{7}{8}} & 42x^4_{n+\frac{7}{8}} \\ 0 & 0 & 2 & 6x^1_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} & 42x^4_{n+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_{n+\frac{1}{8}} \\ y_{n+\frac{3}{8}} \\ f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{5}{8}} \\ f_{n+\frac{7}{8}} \\ f_{n+1} \end{pmatrix} \tag{2.5}$$

Solving for $a'_j s$ in the (2.5) and the resulting value of $a'_j s$ are substituted into (2.1) to yields a continuous implicit hybrid one step method of the form:

$$y(x) = \alpha_{\frac{1}{8}}(t) + \alpha_{\frac{3}{8}}(t) + \beta_0(t) + \beta_{\frac{1}{8}}(t) + \beta_{\frac{3}{8}}(t) + \beta_{\frac{5}{8}}(t) + \beta_{\frac{7}{8}}(t) + \beta_1(t) \tag{2.6}$$

**Where**

$$\left.\begin{aligned}
\alpha_{\frac{1}{8}} &= \frac{3}{2} - 4t \\
\alpha_{\frac{1}{4}} &= -\frac{1}{2} + 4t \\
\beta_0 &= \frac{1}{2400}h^2 - \frac{10597}{282240}t h^2 + \frac{1}{2}t^2 h^2 - \frac{1513}{630}t^3 h^2 + \frac{192}{35}t^4 h^2 - \frac{3424}{525}t^5 h^2 + \frac{2048}{525}t^6 h^2 - \frac{2048}{2205}t^7 h^2 \\
\beta_{\frac{1}{8}} &= \frac{7729}{430080}h^2 - \frac{201253}{1128960}t h^2 + \frac{10}{3}t^3 h^2 - \frac{673}{63}t^4 h^2 + \frac{1528}{105}t^5 h^2 - \frac{2944}{315}t^6 h^2 + \frac{1048}{441}t^7 h^2 \\
\beta_{\frac{3}{8}} &= \frac{1957}{307200}h^2 - \frac{6599}{161280}t h^2 - \frac{14}{9}t^3 h^2 + \frac{137}{15}t^4 h^2 - \frac{1208}{75}t^5 h^2 + \frac{896}{75}t^6 h^2 - \frac{1024}{315}t^7 h^2 \\
\beta_{\frac{5}{8}} &= -\frac{563}{307200}h^2 + \frac{1447}{161280}t h^2 + \frac{14}{15}t^3 h^2 - \frac{269}{45}t^4 h^2 + \frac{952}{75}t^5 h^2 - \frac{2432}{225}t^6 h^2 + \frac{1024}{315}t^7 h^2 \\
\beta_{\frac{7}{8}} &= \frac{47}{61440}h^2 - \frac{3707}{1128960}t h^2 - \frac{10}{21}t^3 h^2 + \frac{199}{63}t^4 h^2 - \frac{152}{21}t^5 h^2 + \frac{2176}{315}t^6 h^2 - \frac{1024}{441}t^7 h^2 \\
\beta_1 &= -\frac{17}{67200}h^2 + \frac{293}{282240}t h^2 + \frac{1}{6}t^3 h^2 - \frac{352}{315}t^4 h^2 + \frac{1376}{1575}t^5 h^2 - \frac{4096}{1575}t^6 h^2 + \frac{2048}{2205}t^7 h^2
\end{aligned}\right\} \tag{2.7}$$

Evaluating (2.6) to obtain the continuous form as,

$$\begin{pmatrix} y_n \\ y_{n+\frac{5}{8}} \\ y_{n+\frac{7}{8}} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} y_{n+\frac{1}{8}} \\ y_{n+\frac{3}{8}} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -1 & 2 \\ -2 & 3 \\ -\frac{5}{2} & \frac{7}{2} \end{pmatrix} + h^2 \begin{pmatrix} \frac{1}{2400} & \frac{7729}{430080} & \frac{1957}{307200} & -\frac{563}{307200} & \frac{47}{61440} & -\frac{17}{67200} \\ -\frac{3}{2800} & \frac{17}{2240} & \frac{119}{2400} & \frac{11}{1600} & -\frac{1}{1680} & \frac{1}{8400} \\ -\frac{17}{8400} & \frac{7}{480} & \frac{509}{4800} & \frac{19}{300} & \frac{43}{6720} & -\frac{1}{1200} \\ -\frac{37}{13440} & \frac{1621}{86016} & \frac{8137}{61440} & \frac{6017}{61440} & \frac{799}{28672} & -\frac{1}{1120} \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{5}{8}} \\ f_{n+\frac{7}{8}} \\ f_{n+1} \end{pmatrix} \tag{2.8}$$

Differentiating (2.6) once, yields

$$y'(x) = \alpha'_{\frac{1}{8}}(t) + \alpha'_{\frac{3}{8}}(t) + \beta'_0(t) + \beta'_{\frac{1}{8}}(t) + \beta'_{\frac{3}{8}}(t) + \beta'_{\frac{5}{8}}(t) + \beta'_{\frac{7}{8}}(t) + \beta'_1(t) \tag{2.9}$$

**Where**

$$
\left.\begin{aligned}
&\alpha'_{\frac{1}{8}}=4\\
&\alpha'_{\frac{3}{8}}=4\\
&\beta'_{0}=-\frac{10597}{282240}h^{2}+t\,h^{2}-\frac{1513}{210}t^{2}h^{2}+\frac{768}{35}t^{3}h^{2}-\frac{3424}{105}t^{4}h^{2}+\frac{4096}{175}t^{5}h^{2}-\frac{2048}{315}t^{6}h^{2}\\
&\beta'_{\frac{1}{8}}=-\frac{201253}{1128960}h^{2}+10\,t^{2}h^{2}-\frac{2692}{63}t^{3}h^{2}+\frac{1528}{21}t^{4}h^{2}-\frac{5888}{105}t^{5}h^{2}+\frac{1024}{63}t^{6}h^{2}\\
&\beta'_{\frac{3}{8}}=-\frac{6599}{161280}h^{2}-\frac{14}{3}t^{2}h^{2}+\frac{548}{15}t^{3}h^{2}-\frac{1208}{15}t^{4}h^{2}+\frac{1792}{25}t^{5}h^{2}-\frac{1024}{45}t^{6}h^{2}\\
&\beta'_{\frac{5}{8}}=\frac{1447}{161280}h^{2}+\frac{14}{5}t^{2}h^{2}-\frac{1076}{45}t^{3}h^{2}+\frac{952}{15}t^{4}h^{2}-\frac{4864}{75}t^{5}h^{2}+\frac{1024}{45}t^{6}h^{2}\\
&\beta'_{\frac{7}{8}}=-\frac{3707}{1128960}h^{2}-\frac{10}{7}t^{2}h^{2}+\frac{796}{63}t^{3}h^{2}-\frac{760}{21}t^{4}h^{2}+\frac{4352}{105}t^{5}h^{2}-\frac{1024}{63}t^{6}h^{2}\\
&\beta'_{1}=\frac{293}{282240}h^{2}+\frac{1}{2}t^{2}h^{2}-\frac{1408}{210}t^{3}h^{2}+\frac{1376}{105}t^{4}h^{2}-\frac{8192}{525}t^{5}h^{2}+\frac{2048}{315}t^{6}h^{2}
\end{aligned}\right\} \tag{2.10}
$$

Evaluating (2.9) at all points, yields

$$
\begin{pmatrix} h\,y'_{n}\\ h\,y'_{n+\frac{1}{8}}\\ h\,y'_{n+\frac{3}{8}}\\ h\,y'_{n+\frac{5}{8}}\\ h\,y'_{n+\frac{7}{8}}\\ h\,y'_{n+1}\end{pmatrix}
=\begin{pmatrix} y_{n+\frac{1}{8}}\\ y_{n+\frac{3}{8}}\end{pmatrix}
\begin{pmatrix} -4 & 4\\ -4 & 4\\ -4 & 4\\ -4 & 4\\ -4 & 4\\ -4 & 4\end{pmatrix}
+h^{2}\begin{pmatrix}
\frac{57287}{967680} & \frac{57287}{967680} & \frac{57287}{967680} & \frac{57287}{967680} & \frac{57287}{967680} & \frac{57287}{967680}\\
\frac{923}{88200} & -\frac{1261}{14112} & -\frac{1513}{25200} & \frac{989}{50400} & -\frac{151}{17640} & \frac{253}{88200}\\
-\frac{701}{88200} & \frac{1513}{35280} & \frac{5297}{50400} & -\frac{131}{6300} & \frac{607}{70560} & -\frac{251}{8820}\\
-\frac{17}{17640} & \frac{1367}{70560} & \frac{1243}{5040} & \frac{1243}{10080} & -\frac{263}{17640} & \frac{73}{17640}\\
-\frac{589}{88200} & \frac{1289}{35280} & \frac{10393}{50400} & \frac{1801}{6300} & \frac{8279}{70560} & -\frac{1259}{88200}\\
-\frac{6841}{1411200} & \frac{35291}{1128960} & \frac{174877}{806400} & \frac{215107}{806400} & \frac{232837}{1128960} & \frac{47609}{1411200}
\end{pmatrix}
\begin{pmatrix} f_{n}\\ f_{n+\frac{1}{8}}\\ f_{n+\frac{3}{8}}\\ f_{n+\frac{5}{8}}\\ f_{n+\frac{7}{8}}\\ f_{n+1}\end{pmatrix} \tag{2.11}
$$

Combining and solving (2.8) and (2.11) simultaneously, yields the explicit schemes as;

$$
\begin{pmatrix} y_{n+\frac{1}{8}}\\ y_{n+\frac{3}{8}}\\ y_{n+\frac{5}{8}}\\ y_{n+\frac{7}{8}}\\ y_{n+1}\end{pmatrix}
=y_{n}+h\,y'_{n}\begin{pmatrix}\frac{1}{8}\\ \frac{3}{8}\\ \frac{5}{8}\\ \frac{7}{8}\\ 1\end{pmatrix}
+f_{n}\begin{pmatrix}\frac{48281}{11289600}\\ \frac{17139}{1254400}\\ \frac{9925}{4541584}\\ \frac{7007}{230400}\\ \frac{379}{11025}\end{pmatrix}h^{2}
+\begin{pmatrix}
\frac{1217}{282240} & -\frac{4051}{3225600} & \frac{1147}{1612800} & -\frac{1601}{4515840} & \frac{1391}{11289600}\\
\frac{4905}{100352} & \frac{201}{22400} & -\frac{549}{358400} & \frac{117}{250880} & -\frac{171}{1254400}\\
\frac{45625}{451584} & \frac{8875}{129025} & \frac{25}{8064} & \frac{625}{903168} & -\frac{125}{451584}\\
\frac{14063}{92160} & \frac{31213}{230400} & \frac{26411}{460800} & \frac{49}{5760} & -\frac{343}{230400}\\
\frac{79}{441} & \frac{263}{1575} & \frac{143}{1575} & \frac{67}{2205} & -\frac{37}{22050}
\end{pmatrix}
\begin{pmatrix} f_{n+\frac{1}{8}}\\ f_{n+\frac{3}{8}}\\ f_{n+\frac{5}{8}}\\ f_{n+\frac{7}{8}}\\ f_{n+1}\end{pmatrix} \tag{2.12}
$$

$$
\begin{pmatrix} y'_{n+\frac{1}{8}}\\ y'_{n+\frac{3}{8}}\\ y'_{n+\frac{5}{8}}\\ y'_{n+\frac{7}{8}}\\ y'_{n+1}\end{pmatrix}
=y'_{n}+f_{n}\begin{pmatrix}\frac{9679}{201600}\\ \frac{663}{22400}\\ \frac{295}{8064}\\ \frac{889}{28800}\\ \frac{103}{3150}\end{pmatrix}h
+\begin{pmatrix}
\frac{14339}{161280} & -\frac{2203}{115200} & \frac{409}{38400} & -\frac{851}{161280} & \frac{41}{22400}\\
\frac{3963}{17920} & \frac{1869}{12800} & -\frac{381}{12800} & \frac{213}{17920} & -\frac{87}{22400}\\
\frac{2125}{10752} & \frac{1325}{4608} & \frac{515}{4608} & -\frac{125}{10752} & \frac{25}{8064}\\
\frac{4949}{23040} & \frac{28469}{115200} & \frac{10633}{38400} & \frac{2779}{23040} & -\frac{49}{230400}\\
\frac{22}{105} & \frac{58}{225} & \frac{58}{225} & \frac{22}{105} & -\frac{103}{3150}
\end{pmatrix}
\begin{pmatrix} f_{n+\frac{1}{8}}\\ f_{n+\frac{3}{8}}\\ f_{n+\frac{5}{8}}\\ f_{n+\frac{7}{8}}\\ f_{n+1}\end{pmatrix} \tag{2.13}
$$

***Properties of the block method***

The analysis of the block method, which includes the order, error constant, consistency, zero stability, convergence and region of absolute stability of the method shall be studied.

***Order and error constant***

Consider the linear operator defined by $\ell\left[y(x);h\right]$, **where**,

$$\Delta\{y(x):h\}=A^{(0)}Y_m^{(i)}-\sum_{i=0}^{k}\frac{jh^{(i)}}{i!}y_n^{(i)}-h^{(3-1)}\left[d_i f(y_n)+b_i F(Y_m)\right], \tag{3.1}$$

Expanding $Y_m$ *and* $F(Y_m)$ in Taylor series and comparing the coefficients of $h$ gives

$$\Delta\{y(x):h\}=C_0 y(x)+C_1 y'(x)+\cdots+C_p h^p y^p(x)+C_{p+1}h^{p+1}y^{p+1}(x)+C_{p+2}h^{p+2}y^{p+2}(x)+\cdots \tag{3.2}$$

**Definition 3.1:** The linear operator $L$ and the associate block method are said to be of order $p$ if $C_0=C_1=\cdots=C_p=C_{p+1}=0,\quad C_{p+2}\neq 0.\quad C_{p+2}$ is called the error constant and implies that the truncation error is given by $t_{n+k}=C_{p+2}h^{p+2}y^{p+3}(x)+0h^{p+3}$

$$L\{y(x):h\}=C_0 y(x)+C_1 y'(x)+\cdots+C_p h^p y^p(x)+C_{p+1}h^{p+1}y^{p+1}(x)+C_{p+2}h^{p+2}y^{p+2}(x)+\cdots \tag{3.3}$$

$$\begin{bmatrix}
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{8}\right)^j}{j!}-y_n-\frac{1}{8}hy'_n-\frac{48281}{11289600}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[-\frac{1217}{282240}\left(\frac{1}{8}\right)+\frac{4051}{3225600}\left(\frac{3}{8}\right)+\frac{1147}{1612800}\left(\frac{5}{8}\right)+\frac{1601}{4515840}\left(\frac{7}{8}\right)-\frac{1391}{11289600}(1)\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{3}{8}\right)^j}{j!}-y_n-\frac{3}{8}hy'_n-\frac{17139}{1254400}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[-\frac{4905}{100352}\left(\frac{1}{8}\right)-\frac{201}{22400}\left(\frac{3}{8}\right)+\frac{549}{358400}\left(\frac{5}{8}\right)-\frac{117}{250880}\left(\frac{7}{8}\right)+\frac{171}{1254400}(1)\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{5}{8}\right)^j}{j!}-y_n-\frac{5}{8}hy'_n-\frac{9925}{451584}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[-\frac{45625}{451584}\left(\frac{1}{8}\right)-\frac{8875}{129024}\left(\frac{3}{8}\right)-\frac{25}{8064}\left(\frac{5}{8}\right)-\frac{625}{903168}\left(\frac{7}{8}\right)+\frac{125}{451584}(1)\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{7}{8}\right)^j}{j!}-y_n-\frac{7}{8}hy'_n-\frac{7007}{230400}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[-\frac{14063}{92160}\left(\frac{1}{8}\right)-\frac{31213}{230400}\left(\frac{3}{8}\right)-\frac{26411}{460800}\left(\frac{5}{8}\right)-\frac{49}{5760}\left(\frac{7}{8}\right)+\frac{343}{230400}(1)\right]\\
\sum_{j=0}^{\infty}\frac{(1)^j}{j!}-y_n-hy'_n-\frac{375}{11025}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[-\frac{79}{441}\left(\frac{1}{8}\right)-\frac{263}{1575}\left(\frac{3}{8}\right)-\frac{143}{1575}\left(\frac{5}{8}\right)-\frac{67}{2205}\left(\frac{7}{8}\right)+\frac{37}{22050}(1)\right]
\end{bmatrix} \tag{3.4}$$

Comparing the coefficient of $h$ in (3.4), according to Skwame *et al*. (2019b) and Sunday (2018), the method is of order $p=4$ and the error constant are given respectively by,

$$C_{p+2}=\left[-6.1409\times10^{-8}\quad 9.1280\times10^{-8}\quad -3.1537\times10^{-8}\quad 1.2115\times10^{-7}\quad 5.9743\times10^{-8}\right]$$

***Consistency of the method***

**Definition 3.2:** According to Dahlquist (1956), a block method is said to be consistent if its order is greater than or equal to one. From the above analysis, it is obvious that our method is consistent.

***Zero stability of the method***

**Definition 3.3:** The numerical method is said to be zero-stable, if the roots $q_s,\ s=1,2,\cdots,k$ of the first characteristics polynomial $\rho(q)$ defined by $\rho(q)=\det\left(qA^{(0)}-E\right)$ satisfies $|q_s|\le 1$ and every root satisfies $|z_s|=1$ have multiplicity not exceeding the order of the differential equation, (Sunday 2018). The first characteristic polynomial is given by,

$$\rho(q)=\left|q\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right|=\left|\begin{bmatrix}q&0&0&0&-1\\0&q&0&0&-1\\0&0&q&0&-1\\0&0&0&q&-1\\0&0&0&0&q-1\end{bmatrix}\right|=q^4(q-1)$$

Thus, solving for $q$ in $q^6(q-1)$ gives $q=0, 0, 0, 0, 1$. Hence, the method is said to be zero stable.

***Convergence of the block method***

**Theorem 3.1:** the necessary and sufficient conditions for linear multistep method to be convergent are that it must be consistent and zero-stable. Hence our method is convergent according to Dahlquist (1956).

***Region of absolute stability of our method***

**Definition 3.4:** the region of absolute stability is the region of the complex $z$ plane, where $z=\lambda h$ for which the method is absolute stable. To determine the region of absolute stability of the block method, the methods that compare neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. Thus, the method according to Sunday (2018) is called the boundary locus method. Applying the method we obtain the region of absolute stability in as;

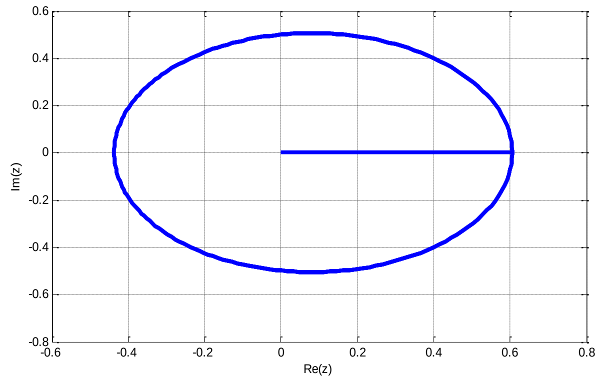


**Fig. 1: Region of absolute stability of our method**

***Numerical implementation of the method***

In this section, we will test the effectiveness and validity of one-step block method by applying on some second order highly stiff problems of the form (1.1) without reduction method. Our result are compared with the existing methods of Omole & Ogunware (2018), Olanegan *et al*. (2018), Skwame *et al*. (2020), Adeniran & Ogundare (2015) and Adeniran *et al*. (2015).

**Problem 4.1: Real-life Problem**

***Cooling of a body***

The temperature $y$ degrees of a body $t$ minutes after being placed in a certain room, satisfies the differential equation $3\frac{d^2y}{dt^2}+\frac{dy}{dt}=0$. By using the substitution $z=\frac{dy}{dt}$ or otherwise, find $y$ in terms of $t$ given that $y=60$ when $t=0$ and $y=35$ when $t=6\ln 4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get?

***Formulation of the Problem***

$$y''=\frac{-y'}{3},\ y(0)=60,\ y'(0)=-\frac{80}{9},\ h=0.1$$

With analytic solution

$$y(x)=\frac{80}{3}e^{-\left(\frac{1}{3}\right)x}+\frac{100}{3}.$$

See Omole & Ogunware (2018); Olanegan *et al*. (2018) and Skwame *et al*. (2020).

**Table 1: Absolute errors for Problem 4.1**

| X | Exact Result | Computed Result | Error in our Method | Error in Omole & Ogunware (2018) | Error in Olanegan *et al.* (2018) | Error in Skwame *et al.* (2020) |
|---|---|---|---|---|---|---|
| 0.1 | 59.12576267952015738700 | 59.12576267952015738700 | 0.0000e-00 | 3.5500e-11 | 7.4764e-06 | 2.3000e-17 |
| 0.2 | 58.28018626750980633900 | 58.28018626750980633500 | 4.0000e-18 | 4.5800e-11 | 2.9394e-05 | 1.7100e-16 |
| 0.3 | 57.46233114762558861700 | 57.46233114762558860800 | 9.0000e-18 | 7.0000e-11 | 6.4802e-05 | 4.3700e-16 |
| 0.4 | 56.67128850781193210600 | 56.67128850781193208900 | 1.7000e-17 | 6.5000e-11 | 1.1279e-05 | 8.1300e-16 |
| 0.5 | 55.90617933041637530700 | 55.90617933041637528100 | 2.6000e-17 | 3.3300e-11 | 1.7250e-04 | 1.2910e-15 |
| 0.6 | 55.16615341541284956400 | 55.16615341541284952600 | 3.8000e-17 | 4.2000e-11 | 2.4310e-04 | 1.8640e-15 |
| 0.7 | 54.45038843564751105000 | 54.45038843564751099900 | 5.1000e-17 | 4.3800e-11 | 3.2383e-04 | 2.5250e-15 |
| 0.8 | 53.75808902305729847200 | 53.75808902305729840700 | 6.5000e-17 | 1.0700e-10 | 4.1393e-04 | 3.2690e-15 |
| 0.9 | 53.08848588484580976200 | 53.08848588484580968100 | 8.1000e-17 | 6.5800e-11 | 5.1271e-04 | 4.0890e-15 |
| 1.0 | 52.44083494863438001100 | 52.44083494863437991400 | 9.7000e-17 | 1.6900e-10 | 6.1951e-04 | 4.9800e-15 |

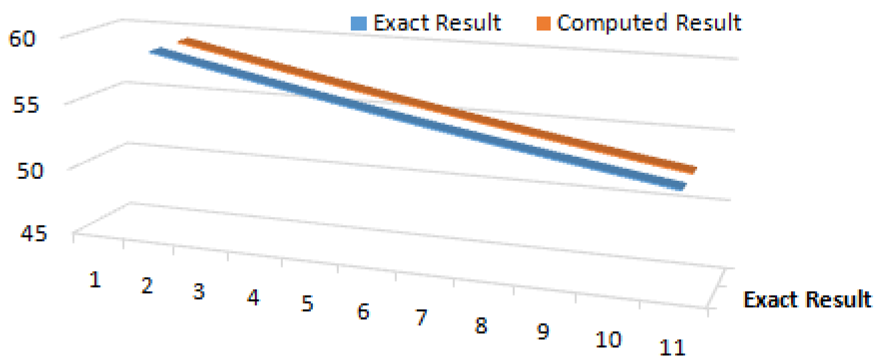


**Fig. 2: Graphical solution of Problem 4.1.**

**Problem 4.2**

Consider a highly stiff linear second order problem

$$y''+100y'+100y=0,\quad y(0)=0,\; y'(0)=-1,\; h=0.1$$

With exact solutions: $y(x)=e^{-x}$.

See Skwame *et al*. (2020), Adeniran & Ogundare (2015), Adeniran *et al*. (2015).

**Table 2: Absolute errors for Problem 4.2**

| X | Exact Result | Computed Result | Error in our Method | Error in Skwame *et al*. (2020) | Error in Adeniran & Ogundare (2015) | Error in Adeniran *et al*. (2015) |
|---|---|---|---|---|---|---|
| 0.1 | 0.904837418035959573l6 | 0.90483741803595952927 | 4.3890e-17 | 3.7209e-15 | 1.0547e-14 | 2.9000e-09 |
| 0.2 | 0.81873075307798185867 | 0.81873075307798182897 | 2.9700e-17 | 8.7829e-14 | 1.7764e-14 | 1.8700e-08 |
| 0.3 | 0.74081822068171786607 | 0.74081822068171781989 | 4.6180e-17 | 1.8840e-12 | 2.3426e-14 | 9.9700e-08 |
| 0.4 | 0.67032004603563930074 | 0.67032004603563925767 | 4.3070e-17 | 4.0785e-11 | 2.7978e-14 | 5.2510e-08 |
| 0.5 | 0.60653065971263342360 | 0.60653065971263337424 | 4.9360e-17 | 8.8239e-10 | 3.1308e-14 | 2.7480e-07 |
| 0.6 | 0.54881163609402643263 | 0.54881163609402638382 | 4.8810e-17 | 1.9092e-08 | 3.3973e-14 | 1.4360e-06 |
| 0.7 | 0.49658530379140951470 | 0.49658530379140946379 | 5.0910e-17 | 4.1306e-07 | 3.5638e-14 | 7.4970e-06 |
| 0.8 | 0.44932896411722159143 | 0.44932896411722154097 | 5.0460e-17 | 8.9370e-06 | 3.6748e-14 | 3.9150e-05 |
| 0.9 | 0.40656965974059911188 | 0.40656965974059906128 | 5.0600e-17 | 1.9336e-04 | 3.7304e-14 | 2.0440e-04 |
| 1.0 | 0.36787944117144232160 | 0.36787944117144227189 | 4.9710e-17 | 4.1836e-03 | 3.7415e-14 | 1.0680e-03 |

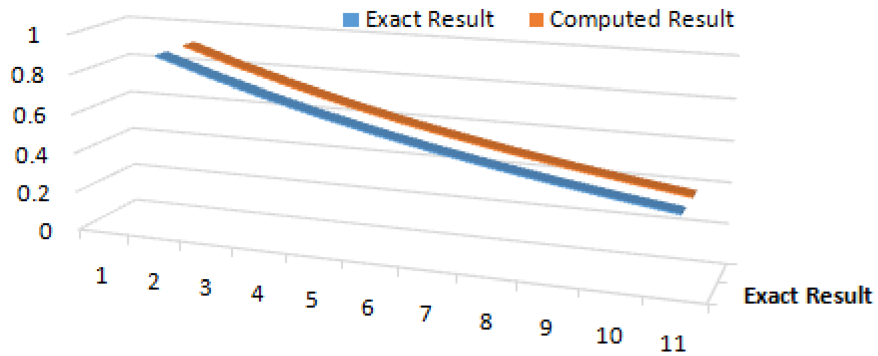


**Fig. 3: Graphical solution of Problem 4.2**

**Discussion of Results and Conclusion**

The developed block method for solvinghigher order ordinary differential equation on implicit one-step second derivative was studied in this research. The method was derived using interpolation and collocate as a basic function. The properties of the one-step block method was analyzed. The method was tested with some numerical examples solved by Omole & Ogunware (2018), Olanegan *et al*. (2018), Skwame *et al*. (2020), Adeniran & Ogundare (2015), Alkasassbeh & Omar (2015) and it is obvious that our method found to give better accuracy when compared. The solution graph shown the convergence of the method in contrast with the exact solutions.

**Conflict of Interest**

The authors declare that there is no conflict of interest related to this work.

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